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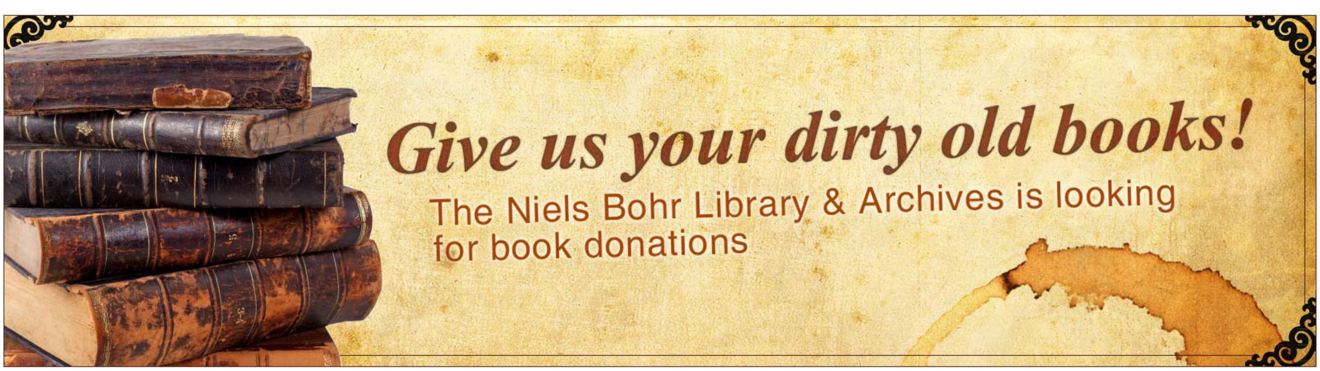

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Soliton-like solutions to the ordinary Schrödinger equation within standard quantum mechanics

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In recent times attention has been paid to the fact that (linear) wave equations admit of "soliton-like" solutions, known as localized waves or non-diffracting waves, which propagate without distortion in one direction. Such localized solutions (existing also for K-G or Dirac equations) are *a priori* suitable, more than gaussian's, for describing elementary particle motion. In this paper we show that, *mutatis mutandis*, localized solutions exist even for the ordinary (linear) Schrödinger equation within standard quantum mechanics; and we obtain both approximate and exact solutions, also setting forth for them particular examples. In the ideal case such solutions (even if localized and "decaying") are not square-integrable, as well as plane or spherical waves: we show therefore how to obtain finite-energy solutions. At last, we briefly consider solutions for a particle moving in the presence of a potential. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4705693]

I. INTRODUCTION

Recently it has been shown—as it had been already realized in old times[1]—that not only nonlinear, but also a large class of linear equations (including, in particular, the wave equations) admit of "soliton-like" solutions. Those solutions[2] are *localized,* and travel along their propagation axis practically without diffracting (at least until a certain field-depth:[2–4]) Such wavelets were indeed called "undistorted progressing waves" by Courant and Hilbert.[1] Let us recall that their peak-velocity V can assume any values[2,5,6] $0 \leq V \leq \infty$, even if we are mainly interested here in their *localization properties* rather than in their peak-velocity. In the case of wave equations, the localized solutions (LSs) easier to be constructed in exact form resulted to be the so-called "(superluminal) X-shaped" ones (see Refs. 2, 4, 7, and 8, and references therein).

The X-shaped waves, long ago predicted[6] to exist within special relativity (SR), have been first mathematically constructed[2,9] as solutions to the wave equations in acoustics,[4] and later on in electromagnetism (namely, to the Maxwell equations[7]), and soon after produced experimentally.[10] Only very recently, *subluminal* localized solutions have been suitably worked out in exact form,[11] even for the case of zero speed ("Frozen Waves").[12]

It was soon thought that, since the mentioned solutions to the wave equations are non-diffractive and particle-like, they may well be related to elementary particles (and to their wave nature).[13,14] And, in fact, localized solutions have been found for Klein-Gordon and for Dirac equations.[13,14]

But little work[15] has been done, as far as we know, for the (*different*) case of the ordinary *Schrödinger* equation.[16] Indeed, the relation between the energy E and the impulse magnitude $p \equiv |\boldsymbol{p}|$ is quadratic [$E = p^2/(2m)$] in the non-relativistic case, like in Schrödinger's, at variance with the relativistic one.

We should mention, however, that many localized (especially X-shaped) solutions have been constructed to the linear[17] or nonlinear[15,18] equation that in optics bear the name of "*Schödinger*

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equation," even if it is mathematically different from the ordinary Schrödinger's. Moreover, a special kind of non-diffracting packet solutions, in terms of Airy functions, were found in the 1970s (Ref. 19) for the case of the actual 1D Schödinger equation, and extended later on to the 3D case.[20] All this has been recently applied to the case of optics, originating the discovery of Airy-type waves, now well known for their remarkable properties:[21] Such Airy waves being solutions, once more, to the so-called (linear) "Schödinger equation" of optics.

But, as we were saying, the nondiffracting solutions, which are essentially superpositions of Bessel beams and are currently called *localized waves*, would be quite apt at describing elementary particles: much more than the gaussian waves. In this paper we show that indeed, *mutatis mutandis*, localized solutions exist even for the *ordinary* (linear) Schrödinger equation within standard quantum mechanics. Our solutions are rather different from the ones found in optics, both for the mentioned fact that the optical Schödinger equation is mathematically different from the ordinary Schrödinger equation, and for the fact that our approach and methods are quite different from the ones adopted in optics.

We are going to obtain both approximate and exact solutions, also setting forth for them particular examples. In the ideal case such solutions, even if localized and "decaying," are not square-integrable (we shall simply say that they bear infinite energy), as well as spherical or plane waves: we shall therefore show how to obtain finite-energy solutions. At last, we shall briefly consider solutions for a particle moving in the presence of a potential.

Before going on, let us recall that, in the time-independent realm—or, rather, when the dependence on time is only harmonic, i.e., for monochromatic solutions—the (quantum, non-relativistic) Schrödinger equation is mathematically identical to the (classical, relativistic) Helmholtz equation.[22] And many trains of localized X-shaped pulses have been found, as *superpositions* of solutions to the Helmholtz equation, which propagate, for instance, along cylindrical or co-axial waveguides;[23] but we shall skip all the cases[24] of this type, even if interesting, since we are concerned here with propagation in free space, even when in the presence of an ordinary potential. Let us also mention that, in the *general* time-*dependent* case, that is, in the case of pulses, the Schrödinger and the ordinary wave equations are no longer mathematically identical, since the time derivative results to be of the fist order in the former and of the second order in the latter. (It has been shown that, nevertheless, at least in some cases,[25] they still share various classes of analogous solutions, differing only in their spreading properties.[25]) Moreover, the Schrödinger equation implies the existence of an *intrinsic* dispersion relation even for free particles, and the result is a temporal spreading even for 1D (one space dimension) pulse solutions.

Let us repeat that the majority of the ideal localized solutions we are going to construct are endowed with infinite energy. We shall treat also a *finite-energy* case (in which the square-integrable solutions travel undistorted and with a constant speed along a *finite* depth of field) only towards the end of this paper: In fact, the infinite-energy solutions themselves, even without truncating them in space and time, result to be quite useful for describing wavepackets in regions not too extended in the transverse direction; as we shall see below.

II. BESSEL BEAMS AS LOCALIZED SOLUTIONS TO THE SCHRÖDINGER EQUATION

Let us consider the Schrödinger equation for a free particle (an electron, for example)

$$\nabla^2\psi + \frac{2im}{\hbar}\frac{\partial\psi}{\partial t} \;=\; 0\,. \tag{1}$$

If we confine ourselves to solutions of the type

$$\psi(\rho, z, \varphi; t) \;=\; F(\rho, z, \varphi)\, \mathrm{e}^{-iEt/\hbar}\,,$$

their spatial part F obeys the reduced equation

$$\nabla^2 F + k^2 F \;=\; 0\,, \tag{2}$$

with $k^2 \equiv p^2/\hbar^2$ and $p^2 = 2mE$ (quantity $p \equiv |\boldsymbol{p}|$ being the particle momentum, and therefore $k \equiv |\boldsymbol{k}|$ the total wavenumber). Equation (2) is nothing but the Helmholtz equation, for which various simple

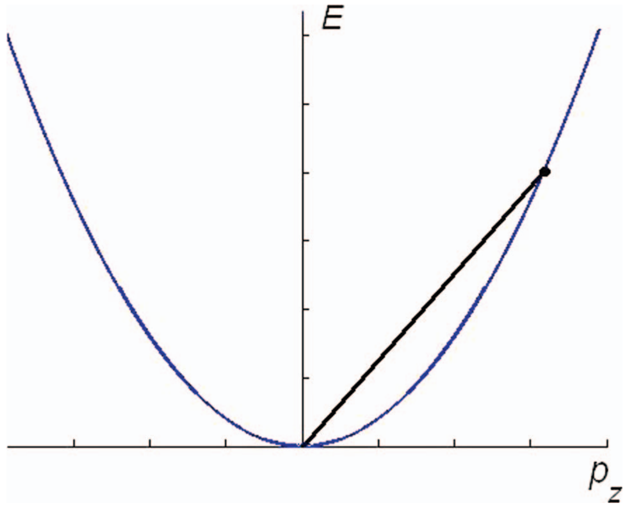


FIG. 1. The allowed region is the one internal to the parabola, since (to avoid divergencies: see the text) it must be $E \geq p_z^2/(2m)$. [The straight-line $E = V p_z$, intersecting the parabola in correspondence with the value $E = 2mV^2$, is here represented for future use.]

localized-beam solutions are already known: In particular, the so-called Bessel beams,[2] which have been experimentally produced since long.[26] Actually, let us look—as usual—for factorized solutions (cylindrically symmetric with respect to [w.r.t.] the z-axis), by putting the constant longitudinal wavenumber to be (if possible) $k_z \equiv p_z/\hbar$. [Since the present formalism is used both in quantum mechanics and in electromagnetism, with a difference in the customary nomenclature, *for clarity's sake let us here stress, or repeat,* that $k \equiv p/\hbar$; $k_\rho \equiv k_\perp \equiv p_\perp/\hbar$; $\omega \equiv E/\hbar$; while $k_z \equiv k_\parallel = p_\parallel/\hbar \equiv p_z/\hbar$ is often represented by the (for us) ambiguous symbol β.] As a consequence, the (transverse) wavefunction obeys a Bessel differential equation, in which it enters the constant transverse wavenumber $k_\rho \equiv p_\rho/\hbar$ with the condition

$$k_\rho^2 = k^2 - k_z^2 \equiv 2mE/\hbar^2 - k_z^2 \; . \tag{3}$$

To avoid any divergencies, it must be $k_\rho^2 \geq 0$, that is, $k^2 \geq k_z^2$; namely, it must hold [see Fig. 1] the constraint

$$E \;\geq\; \frac{p_z^2}{2m} \; .$$

The solution is therefore [$p \equiv \hbar k$]:

$$\psi(\rho, z; t) \;=\; J_0(\rho p_\rho/\hbar) \; \exp\left[i(zp_z - Et)/\hbar\right] \tag{4}$$

together with condition (3). *Equation* (4) *can be regarded as a Bessel beam solution to the Schrödinger equation* [the other Bessel functions are not acceptable here, because of their divergence at $\rho = 0$ or for $\rho \to \infty$] with forward propagation (i.e., positive z direction) for $k_z > 0$. This result is not surprising, since—once we suppose the whole time variation to be expressed by the function $\exp\left[-i\omega t\right]$—both the ordinary wave equation and the Schrödinger equation transform into the Helmholtz equation. Actually, the only difference between the Bessel beam solutions to the wave equation and to the Schrödinger equation consists in the different relationships among frequency, longitudinal, and transverse wavenumber; in other words (with $E \equiv \omega\hbar$):

$$p_\rho^2 \;=\; E^2/c^2 - p_z^2 \quad \text{for the wave equation;} \tag{5a}$$

$$p_\rho^2 \;=\; 2mE - p_z^2 \qquad \text{for the Schodinger equation.} \tag{5b}$$

In the case of beams, the experimental production of LSs to the Schrödinger equation can be similar to the one exploited for the LSs to the wave equations (e.g., in optics, or acoustics): cf., e.g., Figure 1.2 in the first one of Ref. 8, and references therein, where the simple case of a source consisting in an array of circular slits, or rings, was considered.[27] In the table we refer to a Bessel beam of photons, and a Bessel beam of, e.g., electrons, respectively. We list therein the relevant quantities having a role, e.g., in electromagnetism, and the corresponding ones for the Schrödinger equation's spatial part $\hbar^2\nabla^2 F + 2mE\,F = 0$, with $F = R(\rho)\,Z(z)$. The second and the fourth lines have been written down for the simple Durnin *et al.*'s case, when the Bessel beam is produced by an annular slit (illuminated by a plane wave) located at the focus of a lens.[26]

Wave equation	Schrödinger equation
$k \;=\; \frac{\omega}{c}$	$p \;=\; \sqrt{2mE}$
$k_\rho \;\simeq\; \frac{r}{f}\,k$	$p_\rho \simeq \frac{r}{f}\,p$
$k_\rho^2 \;=\; \frac{\omega^2}{c^2} - k_z^2$	$p_\rho^2 \;=\; 2mE - p_z^2$
$k_z^2 \;=\; \frac{\omega^2}{c^2}(1 - \frac{r^2}{f^2})$	$p_z^2 \;=\; 2mE(1 - \frac{r^2}{f^2})$

In this table, quantity *f* is the focal distance of the lens (for instance, an ordinary lens in optics; and a magnetic lens in the case of Schrödinger charged wavepackets), and *r* is the radius of the considered ring. [In connection with the last line of the table, let us recall that in the wave equation case the phase-velocity ω/k_z is almost independent of the frequency (at least for limited frequency intervals, like in optics), and one gets a constant group-velocity and an easy way to build up X-shaped waves. By contrast, in the Schrödinger case, the phase-velocity of each (monochromatic) Bessel beam depends on the frequency, and this makes it difficult to generate an "X-wave" (i.e., a wave depending on z and t only via the quantity $\zeta \equiv z - Vt$) by using simple methods, as Durnin *et al.*'s, based on Bessel beams superposition. In the case of charged particles, one should compensate such a velocity variation by suitably modifying the focal distance *f* of the Durnin's lens, e.g., on having recourse to an additional magnetic, or electric, lens.]

Before going on, let us stress that one could easily eliminate the restriction of axial symmetry: In such a case, in fact, solution (4) would become

$$\psi(\rho, z, \varphi; t) \;=\; J_n(\rho p_\rho/\hbar)\mathrm{e}^{izp_z/\hbar}\,\mathrm{e}^{-iEt/\hbar}\,\mathrm{e}^{in\varphi}\;,$$

with n an integer. The investigation of not cylindrically symmetric solutions is interesting especially in the case of localized *pulses* (cf. Sec. III): and we shall deal with them below.

III. EXACT LOCALIZED SOLUTIONS TO THE SCHRÖDINGER EQUATION (FOR ARBITRARY FREQUENCY SPECTRA)

Localized (non-dispersive, besides non-diffracting) *pulses* can be constructed, as solutions to the Schrödinger equation, both by having recourse to the standard "paraxial approximation," and in an exact, analytic way. Let us start with the analytic method.

Our aim is to construct analytical solutions to the Schrödinger equation, by following an exact approach. Then, let us go back to Eq. (1), and to its Bessel-beam solution (4), where, as before, relation (5b) holds: $p_\rho = \sqrt{2mE - p_z^2}$, with $E = \omega\hbar$ [and one may confine himself to $p_\rho > 0$].

The *condition*[2] for obtaining localized solutions is that

$$E \;=\; Vp_z + b, \tag{6a}$$

with b a positive constant (bearing the appropriate dimensions of an energy, and regulating the position of the chosen straight-line in the plane (E, p_z)); which corresponds in particular, on using Eq. (5b), to the maximum (E_+) and minimum (E_-) values of the energy:

$$E_\pm \;=\; mV^2\left(1 \pm \sqrt{1 + \frac{2b}{mV^2}}\right) + b. \tag{6b}$$

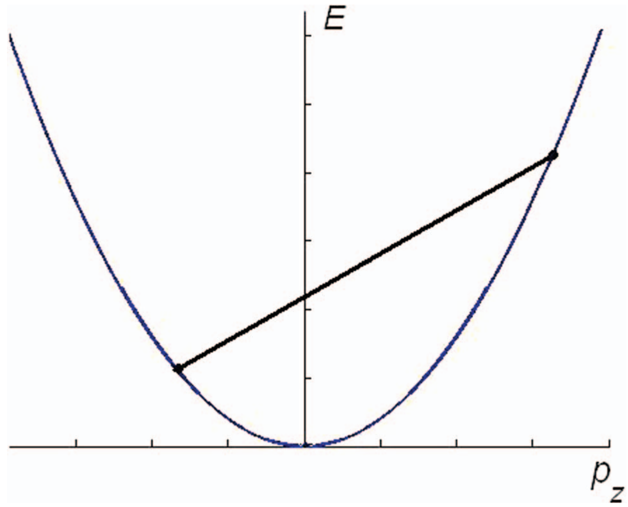


FIG. 2. This time, the parabola and the chosen straight-line have equations $E = p_z^2/(2m)$ and $E = Vp_z + b$, respectively. The intersections of this straight-line with the parabola are now two, whose corresponding values are given in Eq. (6b). Inside the parabola $p_\rho^2 \geq 0$.

The linear constraint (6a) is depicted in Fig. 2; while Fig. 1 presents it in the simple case $b = 0$, for future use.

Localized solutions can therefore be obtained by the following superpositions (integrations over the frequency, or the energy) of Bessel-beam solutions:

$$\Psi(\rho, z, \zeta) \;=\; \mathrm{e}^{\frac{-ib}{\hbar V} z} \int_{E_-}^{E_+} \mathrm{d}E \; J_0(\rho p_\rho/\hbar)\; S(E)\; \mathrm{e}^{i\frac{E}{\hbar V}\zeta} \;, \tag{7}$$

together with

$$p_\rho \;=\; \frac{1}{V}\sqrt{-E^2 + (2mV^2 + 2b)E - b^2}\;. \tag{8}$$

Notice that the in Eq. (7) [as well as in Eq. (12) below], the solution Ψ depends on z, besides via $\zeta \equiv z - Vt$, only via a phase factor; the modulus $|\Psi|$ of Ψ goes on depending on z (and on t) only through the variable ζ.

A. Particular exact localized solutions

We want now to re-write the integral $\mathcal{I}$ appearing in the r.h.s. of Eq. (7) so that its integration limits are -1 and $+1$, respectively; that is, in the form

$$\mathcal{I} \;=\; \int_{-1}^{1} \mathrm{d}u\; S(u)\; J_0(\frac{\rho\sqrt{P}}{\hbar}\sqrt{1-u^2})\; \mathrm{e}^{i\, f(\zeta)\, u},$$

quantity $f(\zeta)$ being an arbitrary dimensionless function. To obtain this, we have to *look for* a transformation of variables [with A and B constants, with the correct dimensions of energy, to be determined]

$$E \;=\; Au + B \tag{9}$$

such that

$$p_\rho^2 \;=\; P(1-u^2)\;; \qquad u_+ = 1; \qquad u_- = -1, \tag{9'}$$

P being a suitable constant (with the appropriate dimensions of a square impulse). On writing $V^2 p_\rho^2 \;=\; E\,(\hbar V^2 M - E) - b^2$, with $\hbar M \equiv 2m + 2b/V^2$, after some algebra one finds that it must be

$$A \;=\; \sqrt{P}\;V\;;\qquad B \;=\; mV^2+b\;;\qquad P \;=\; m^2V^2+2mb\;. \tag{10}$$

Indeed, one can verify (by some more algebra) that Eqs. (9)–(10) imply, as desired, that $u_- = -1$ and $u_+ = 1$.

In conclusion, the transformation

$$E \;=\; mV^2\sqrt{1+\frac{2b}{mV^2}}\;u+mV^2+b \tag{11}$$

does actually allow writing solution (7) in the form [recall that $E = Au + B \Rightarrow \mathrm{d}E = A\mathrm{d}u$]

$$\Psi(\rho,\eta,\zeta) \;=\; \mathcal{N}\,A\,\mathrm{e}^{\frac{imV}{\hbar}\eta}\int_{-1}^{1}\mathrm{d}u\;S(u)\;J_0(\frac{\rho}{\hbar}\sqrt{P}\sqrt{1-u^2})\;\mathrm{e}^{\frac{iA\zeta}{\hbar V}u}\;, \tag{12}$$

with

$$\eta \equiv z - vt\;,$$

where $v \equiv V + b/(mV)$. Equation (12) is exactly, analytically integrable when S is a constant or a suitable exponential.

Let us choose the complex exponential function (which will easily enter as an element in a Fourier expansion)

$$\overline{S}(E) \;=\; a_n\,\mathrm{e}^{\frac{2\pi i}{D}nE}\;, \tag{13}$$

with n an integer, and $D \equiv E_+ - E_- = 2mV^2\sqrt{1+2b/(mV^2)}$, while a_n are constant quantities (with the appropriate dimensions of the inverse of an energy). On remembering that $E = Au + B$, such a spectrum can be written in terms of u as

$$\overline{S}(u) \;=\; a_n\,\mathrm{e}^{i\pi nu}\,\mathrm{e}^{i\frac{2\pi}{D}nB} \tag{13$'$}$$

(still with the dimensions of an inverse energy). After some more algebra, the analytic exact solution to the Schrödinger equation, corresponding to spectrum (13′), results to be[11]

$$\Psi(\rho,\eta,\zeta) \;=\; \mathcal{N}a_n\,2A\,\frac{\sin Z}{Z}\,\mathrm{e}^{\frac{imV}{\hbar}\eta}\,\mathrm{e}^{i\frac{2\pi}{D}nB}\;, \tag{14}$$

where A, B, P are given by Eqs. (10) and

$$Z \;\equiv\; \sqrt{\left(\frac{A}{\hbar V}\zeta+n\pi\right)^2+\frac{P}{\hbar^2}\rho^2}\;. \tag{15}$$

Equation (14), as we have just seen, is a particular exact localized solution to the Schrödinger equation; but we are going to utilize it essentially as *an element* of suitable superpositions. Before going on, however, we wish to depict in Fig. 3 an elementary solution: namely, the square magnitude of the simple solution corresponding, in Eq. (7), to the *real* exponential

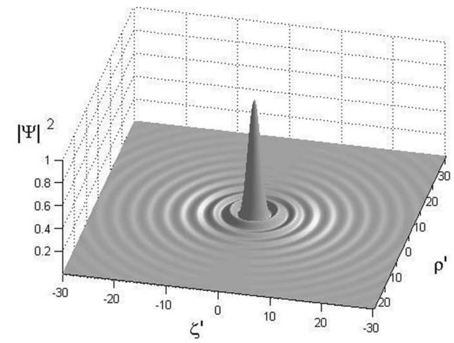
$$S(E) \;=\; s_0\,\exp[a(E-E_+)]\;, \tag{16}$$

a being a positive number, endowed with the appropriate dimensions of an inverse energy, as well as s_0. When $a = 0$, one ends up with a solutions similar to Mackinnon's.[28] Spectrum (36) is exponentially concentrated in the proximity of E_+, where it reaches its maximum value; and becomes more and more concentrated (on the left of E_+, of course) as the arbitrarily chosen value of a increases. To perform the integration in Eq. (7), it is once more useful to operate the variable transformation (9) and go on to Eq. (12), spectrum (16) assuming now the form

$$S(u) \;=\; s_0\,e^{-aE_+}\,e^{aB}\,e^{aAu}\;.$$

Performing the integration in Eq. (12), by a process similar to the one which led us to Eq. (14), in the present case we get

$$\Psi(\rho,\eta,\zeta) \;=\; \mathcal{N}s_0 2V\sqrt{P}\,\exp[i\frac{mV}{\hbar}\eta]\exp[-aV\sqrt{P}]\frac{\sin Y}{Y}, \tag{17a}$$



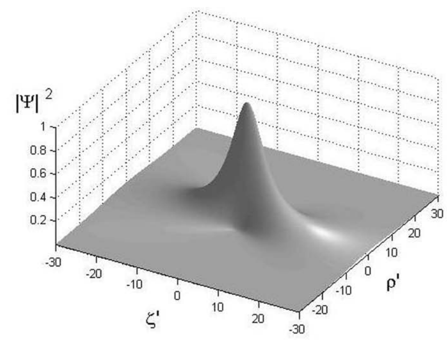


FIG. 3. In these figures we depict an elementary solution: Namely, the square magnitude of the simplified solution, Eq. (17a), corresponding to the *real* spectrum $S(u) = s_0\exp\,[(E - E_+)a]$, as a function of $\rho' \equiv \rho\sqrt{P}/\hbar$ and of $\zeta' \equiv \zeta\sqrt{P}/\hbar$. Quantity a is a positive number [when $a = 0$ one ends up with a solutions similar to Mackinnon's,[28]] while b for simplicity has been chosen equal to zero. (a) corresponds to $a = E_+/5$, while (b) corresponds to $a = 5E_+$. For the properties of the spectral function (16), see the text.

where

$$Y \;\equiv\; \frac{\sqrt{P}}{\hbar}\sqrt{\rho^2 - (\hbar a V + i\zeta)^2}\;, \tag{17b}$$

quantity P having been defined in Eq. (10); and one should remember that $\eta \equiv z - vt$ is a function of b.

Equations (17) appear to be the simplest closed-form nondiffracting solution (see Fig. 3) to the Schrödinger equation, since they do not need any recourse to series expansions of the type exploited in Sec. III B. However, the solutions that we shall construct below can correspond to spectra more general than (16); for instance, to the gaussian spectrum, which possesses two advantages w.r.t. spectrum (16): it can be easily centered around any value of u, that is, around any value $\bar{E}$ of E in the interval $[E_-, E_+]$, and, when increasing its concentration in the surrounding of $\bar{E}$, its "spot" transverse width does not increase indefinitely, at variance with what happens for spectrum (16). Anyway, the exact solutions (17) are noticeable, since they are really the simplest nondiffracting ones.

Some physical (interesting) comments on the results in Eqs. (17) and Fig. 3 will appear elsewhere. Here, let us add only a few further figures and some brief comments. Let us first recall that, as predicted in the first one of Ref. 6, the localized (nondiffracting) solutions to the ordinary wave equations resulted to be roughly *ball-like* when their peak-velocity is subluminal,[11] and *X-shaped*[4,7] when superluminal.

Now, normalizing ρ and ζ, we can write Eq. (17b) as

$$Y \;=\; \sqrt{\rho'^2 - (\overline{A} + i\zeta')^2}$$

with $\rho' \equiv \sqrt{P}\rho/\hbar$ and $\zeta' \equiv \sqrt{P}\zeta/\hbar$, quantity P being given by the last one of Eqs. (10), namely, $P = m^2V^2 + 2mb$, while $\overline{A} \equiv aA = \sqrt{P}aV$. For simplicity, let us confine ourselves to the case $b = 0$, forgetting now about the more interesting cases with $b \neq 0$; therefore, it will hold the simple relation

$$\overline{A} = maV^2\;.$$

In the present case of the Schrödinger equation, we can observe the following.

If we choose $\overline{A} = 0$, which can be associated with $V = 0$, we get the solutions in Fig. 4: that is, a ball-like structure.

By contrast, if we increase the value of $\overline{A}$, by choosing, e.g., $\overline{A} = 20$ (which can be associated with larger speeds), one notices that also a X-shaped structure starts to contribute: see, e.g., Fig. 5.

To have a preliminary idea of the "internal structure" of our soliton-like solutions to the (ordinary) Schrödinger equation, let us plot, instead of the square magnitude of Ψ, its real or imaginary part:

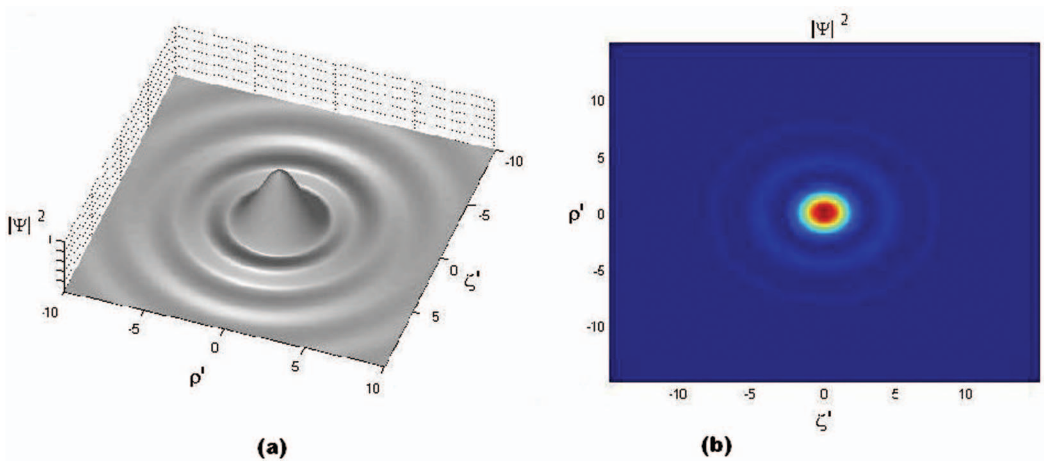


FIG. 4. In these, and the following Figs. 5–7, we depict the square magnitude of some more solutions of the type (17a), normalized with respect to ρ and ζ; still assuming for simplicity $b = 0$, so that $\overline{A} = maV^2$. The present figures show the "ball-like" structure that one gets, as expected, when $\overline{A} = 0$ (see the text, also for the definitions of ρ' and ζ'). (b) The projection on the plane (ρ', ζ') of the 3D plot shown in (a).

Let us choose its real part, or rather the square of its real part. Then even in the $\overline{A} = 0$ case one starts to see the appearance of the X shape, which becomes more and more evident as the value of $\overline{A}$ increases: In Fig. 6 we show the projections on the plane (ζ', ρ') of the real-part square for the solutions with $\overline{A} = 5$ and $\overline{A} = 50$, respectively. Further attention to such aspects will be paid elsewhere (including, for instance, also studies about the angular momentum or perturbation theory of such solutions).

But the (square of the) real part of Ψ does show, in 3D, also some "internal oscillations": cf., e.g., Fig. 7 corresponding to the value $\overline{A} = 5$. As we said, we shall face elsewhere, however, topics like their possible connections with the de Broglie picture of quantum particles, et alia.

B. A general exact localized solution

Let us go back to our spectrum $S(E)$ in Eq. (13). Since in our fundamental equation (7) the integration interval is limited $[E_- < E < E_+]$, in such an interval *any* spectral function $S(E)$ whatever

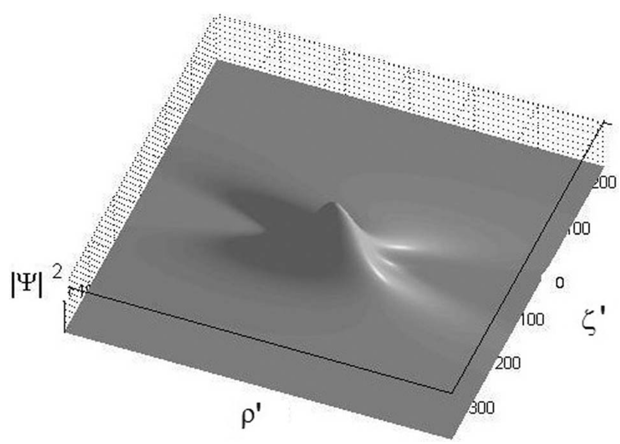


FIG. 5. The solution, under all the previous conditions, with an increased value of $\overline{A}$, namely with $\overline{A} = 20$. An X-shaped structure starts to appear, contributing to the general form of the solution (see the text).

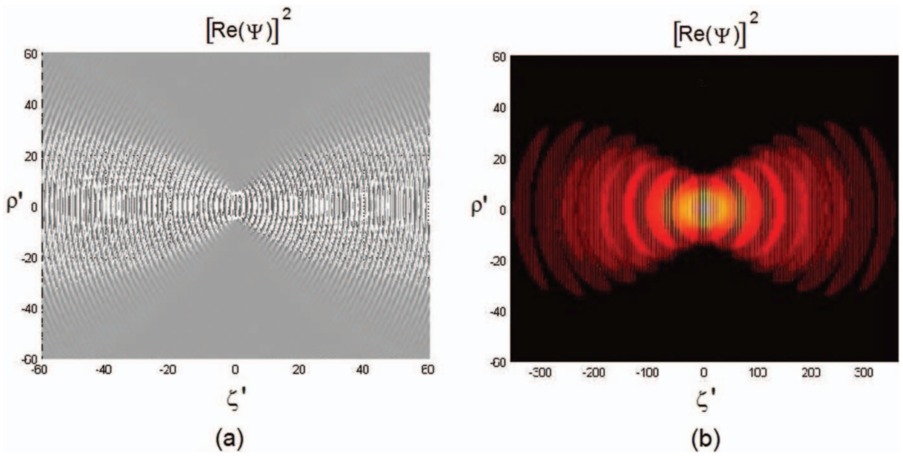


FIG. 6. To get a preliminary idea of the "internal structure" of our soliton-like solutions, it is useful to have recourse (see the text) to the real part of Ψ. In these figures we plot the projections on the plane (ζ', ρ') of the real-part square for the solutions with $\overline{A} = 5$ (a) and $\overline{A} = 50$ (b), respectively [once more, we are using coordinates ζ, ρ *normalized* as defined in the text].

can be expanded into the Fourier series

$$S(E) \;=\; \sum_{n=-\infty}^{\infty} a_n \, \mathrm{e}^{i \frac{2\pi}{D} n E} \tag{18}$$

with

$$a_n \;=\; \frac{1}{D} \int_{E_-}^{E_+} \mathrm{d}E \; S(E) \, \mathrm{e}^{-i \frac{2\pi}{D} n E} \;, \tag{19}$$

quantity $S(E)$ being an *arbitrary* function, and D being still defined as $D \equiv E_+ - E_-$.

Inserting Eq. (18) into Eq. (7), and following the same procedure exploited in Sec. III A (in particular, going on again from E to the new variable u), we end up with the *general exact localized*

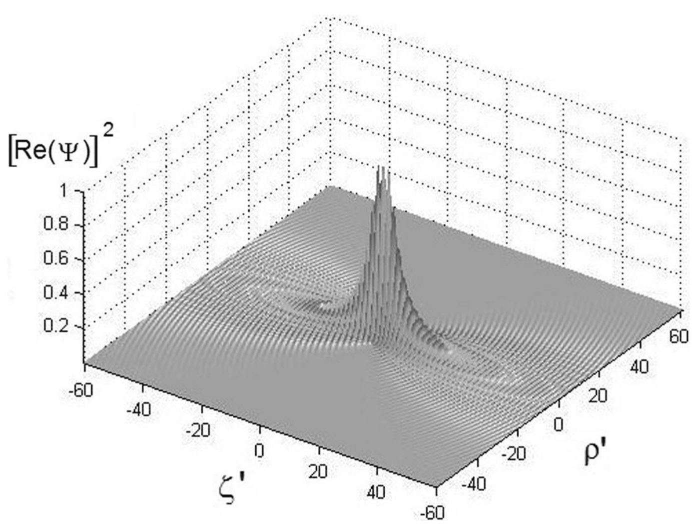


FIG. 7. The (square of the) real part of Ψ shows, in 3D, also some "internal oscillations": this figure corresponds, e.g., to the value $\overline{A} = 5$.

solution to the Schrödinger equation:

$$\Psi(\rho,\eta,\zeta) \;=\; \mathcal{N}\, 2A\, e^{i\frac{mV}{\hbar}\eta} \sum_{n=-\infty}^{\infty} a_n \exp\,[i\frac{2\pi}{D}nB]\;\frac{\sin Z}{Z}\;, \tag{20}$$

where Z is defined in Eq. (15), and the coefficients a_n are given by Eq. (19). The normalization constant $\mathcal{N}$ is chosen so that the maximum value of $|\Psi|^2$ be equal to the unity.

It is worthwhile to note that, even when truncating the series in Eq. (20) at a certain value $n = N$, the solutions obtained is *still* an exact Localized Solution of the Schrödinger equation!

IV. ABOUT SQUARE-INTEGRABLE LOCALIZED SOLUTIONS TO THE SCHRÖDINGER EQUATION

The solutions found above, even if very instructive, are ideal solutions which are not square-integrable; and cannot be accepted in QM. It is important, therefore, to show how to construct finite-energy (i.e., square-integrable) solutions.

Let us obtain localized solution to the Schrödinger equation endowed with *finite energy,* by starting from Eqs. (17). First of all, one has to integrate over b by adopting a spectrum $S(b)$ strongly bumped around a value b_0: We already know, indeed, that spectra of this type are required in order to get solutions that are non-diffracting all along a certain field-depth.

Then, it can be easily seen that the finite-energy solution, $\Psi_{\rm fe}$, can be preliminarily written as

$$\Psi_{\rm fe} \;=\; \mathcal{N}\frac{s_0 V\sqrt{P}}{iY}\,(I_- - I_+)\;, \tag{21}$$

where I_- and I_+ are two (dimensionless) integrations over b from 0 to infinity (quantity b having been defined in Eq. (6a), and therefore having the correct dimensions of an energy), while s_0 appears in Eq. (16).

Let us now pass from b, defined in Eq. (6a), to the new variable $w \equiv \sqrt{P}$. One has to choose a spectrum $S(w)$ corresponding to a $S(b)$ concentrated around a specific value of b; let us therefore adopt the gaussian function

$$\mathcal{S}(w) \;=\; \frac{m\sqrt{q}}{\sqrt{\pi}\,\hbar w}\;\exp[-q(w-w_0)^2]\;, \tag{22}$$

with $w_0 > mV > 0$.

When we go on from b to the new variable $w \equiv \sqrt{P}$ (where P depends on b), the two quantities I_- and I_+ become integrations over w from mV to ∞. After further calculations, and using relation 3.322.1 in Ref. 29, one obtains that

$$I_\pm \;=\; \frac{\sqrt{q}}{U}\,e^{-qw_0}\,e^{\frac{imV}{2\hbar}z}\,\exp\,[\frac{W_\pm^2}{U^2}]\left[1-\Phi\left(\frac{W_\pm}{U}+\frac{mV}{2}U\right)\right]\;, \tag{23}$$

where

$$U \equiv 2\sqrt{q+\frac{i\hbar}{2m}t}\;;\qquad W_\pm \equiv -2qw_0 + aV \pm i\frac{Y}{\sqrt{P}}\;,$$

quantity Y having been defined in Eq. (17b).

We have therefore shown that realistic (finite-energy) localized solutions exist also to the Schrödinger equation; they will be non-diffracting only till a certain finite distance (depth of field). The analysis of explicit, particular examples will be presented elsewhere.

V. THE CASE OF NON-FREE PARTICLES

Let us consider now the case of a particle in the presence of a potential: for simplicity, let us confine ourselves to the case of a *cylindrical potential.*

Namely, let us consider the Schrödinger equation with a potential of the type $U(\rho)$:

$$-\frac{\hbar^2}{2m}\left(\nabla_\perp^2+\frac{\partial^2}{\partial z^2}\right)\psi+U(\rho)\psi-i\hbar\frac{\partial\psi}{\partial t} \;=\; 0. \tag{24}$$

Now, we can use the method of separation of variables writing $\psi = R(x, y)Z(z)T(t)$. With this, we get the well known solutions

$$T = e^{-\frac{i}{\hbar}Et}, \tag{25}$$

$$Z = e^{ip_z z/\hbar}, \tag{26}$$

and the eigenvalue equation

$$-\hbar^2\,\nabla_\perp^2 R + 2m\,U(\rho)\,R \;=\; \Lambda^2 R \tag{27}$$

with

$$\Lambda^2 \;=\; 2mE - p_z^2. \tag{28}$$

Supposing a potential $U(\rho)$ that only allows transverse bound states (as the parabolic potential), we will find the eigenfunctions $R_n(x, y)$ and discrete eigenvalues Λ_n^2.

We can construct more general solutions

$$\Psi = \sum_n f_n R_n(x,y)e^{ik_z z/\hbar}e^{-\frac{i}{\hbar}Et} \tag{29}$$

with

$$2mE \;=\; p_z^2+\Lambda_n^2. \tag{30}$$

Considering $p_z \geq 0$ (forward propagation), the constraint (30) defines a set of parabolas (something like the modes in a waveguide: cf. Ref. 16). Chosen a certain Λ_n^2, once a value for p_z is given, the value of E gets fixed.

To obtain from (29) a train of localized pulses, i.e., a wavefunction $\Psi(x, y, z - Vt)$, we must have

$$E \;=\; Vp_z. \tag{31}$$

So, from conditions (30) and (31), p_z must assume the values

$$p_z \;=\; mV\left(1\pm\sqrt{1-\frac{1}{m^2V^2}\Lambda_n^2}\right) \tag{32}$$

with

$$\Lambda_n \leq mV. \tag{33}$$

Figure 8 illustrates the situation. The values to E and p_z that furnish localized pulse trains are given by the intersection between the parabolas defined by Eq. (30) and the straight-line defined by Eqs. (31). Note that in these cases the series (29) will be always truncated (finite number of terms), due the condition (33). We also have to note that, for any given Λ_N^2, one gets two possible values of p_z (see Eq. (32)), as it can be observed from Fig. 8, in which the straight-line cuts each parabola twice.

For our purpose, the superposition has to be

$$\Psi(x,y,z-Vt) \;=\; \sum_n f_n R_n(x,y)e^{ip_{zn}(z-Vt)/\hbar} \tag{34}$$

with

$$p_z \;=\; mV\left(1\pm\sqrt{1-\frac{1}{m^2V^2}\Lambda_n^2}\right) \tag{35}$$

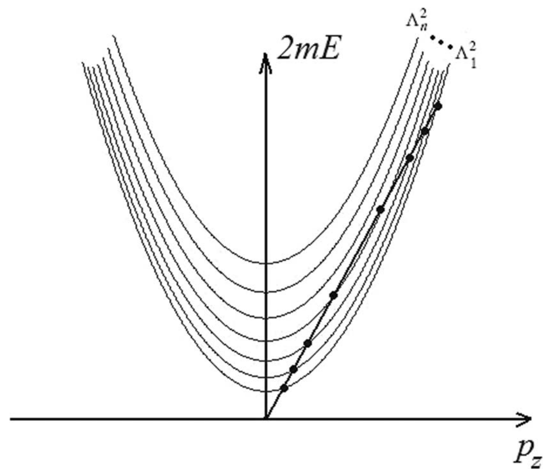


FIG. 8. In the case of a particle in the presence of a cylindrical potential, the values to E and p_z that furnish localized pulse trains are given by the intersections between the parabolas in Eq. (30) and the straight-line in Eq. (31): see the text. It can be noticed that, for any given Λ_n^2, one gets two possible values of p_z (cf. Eq. (32)), since the straight-line cuts each parabola twice. See the text, and cf. also Ref. 23.

and

$$\Lambda_n \leq mV. \tag{36}$$

In principle, any set of coefficients f_n will furnish *trains of localized waves*.

A. Observation

If we look for a square-integrable wave function, we can start from superposition (29) and integrate its terms over p_z around each p_{zn}, respectively (as we already did in our papers on X-type pulses propagating along waveguides[23]). But in the present case, in general, the group-velocities defined at the points p_{zn} will *not* be the same, as it happened in the waveguide case; and we will therefore meet a kind of intermodal dispersion, besides the group-velocity dispersion. Let us recall, incidentally, that such an intermodal dispersion did not occur in the case of X-type waves, traveling in metallic waveguides, due the peculiar fact that the group-velocities defined at those points were always the same). After the integration, we can obtain an envelope with a train of pulses (or just one pulse) inside it. The envelope will suffer dispersion, but the train of pulses inside it will not.

More general localized wave trains can be obtained using the relation $E = Vp_z + b$, with b a positive constant.

In the case of potentials like $U(\rho)$, one can search for solutions with cylindrical symmetry, for simplicity. However, also solutions without this symmetry can be investigated: and they will be interesting for an analysis of the angular momentum.

VI. SOME CONCLUSIONS

Nondiffracting solutions, called localized waves, are known to exist for any (linear) wave equations. In this paper, we have shown that similar "soliton-like" solutions do exist even for the rather different case of the ordinary (linear) Schrödinger equation within standard quantum mechanics. We constructed such solutions, in exact form, both for the ideal case of ("decaying", but) not square-integrable pulses, and for the realistic case of square-integrable pulses, without forgetting the non-free particle cases; and always setting forth appropriate particular examples. In the Appendix below we construct them also in the so-called paraxial approximation, since we can work out therein some simplified (approximate) solutions.

Our new solutions can have a relevant *theoretical* meaning, as mentioned in the text above, since they are *a priori* well suited for the description of elementary particles and quantum objects much more than the ordinary solution, such as the gaussian ones.

From the point of view of their direct production, in the case, e.g., of wavepacket beams the experimental setup for constructing localized solutions to the Schrödinger equation can be similar to the one already used for the (linear) wave equations of optics, when a simple source consisted in an array of circular slits. [Only for pulses the generation technique should deviate from optics', since in the Schrödinger equation case the phase of the Bessel beams produced through one annular slit would depend on the energy.] In the table above (Sec. II) we refer to a Bessel beam of photons, and a Bessel beam of, e.g., electrons, respectively: we listed therein the relevant quantities having a role, e.g., in electromagnetism, and the corresponding ones for the Schrödinger equation's spatial part $\hbar^2\nabla^2 F + 2mE\, F = 0$, with $F = R(\rho)\, Z(z)$. The second and the fourth lines have been written down for the quite simple case of a Bessel beam produced by a single annular slit (illuminated by a plane wave) located at the focus of a lens. To be more specific, quantity f therein is the focal distance of the lens, which can be an ordinary lens in optics, and a *magnetic lens* in the case of Schrödinger charged wavepackets); while r is the radius of the considered ring. Some attention has of course to be paid to the obvious differences between the two contexts. We have already warned, in connection with the last line of the table, that in the wave equation case the phase-velocity ω/k_z is almost independent of the frequency (at least for limited frequency intervals, like in optics), so that one gets a constant group-velocity and an easy way to build up the particular LWs called X-shaped waves. By contrast, in the Schrödinger case, the phase-velocity of each (monochromatic) Bessel beam depends on the frequency, and this makes it more difficult to generate a so-called "X-wave" (i.e., a wave depending on z and t only via the quantity $z - Vt$) by using simple methods, as Durnin *et al.*'s, based on Bessel beams superposition. But in the case of charged particles one could compensate such a velocity variation by suitably modifying the focal distance f of the lens, e.g., on having recourse to an additional magnetic, or electric, lens.

An interesting problem, still from the experimental point of view, is that in optics one starts usually from a laser source. In the case of quantum mechanics, one should have recourse to "laser beams of particles," as the ones under investigation since more than a decade: see, e.g., Ref. 30.

As to further extensions of our approach, allowing a spatio-temporal control of carrier dynamics, let us observe that—as kindly remarked by the referee—even an interplay of photons and electrons in surface plasmon-polaritons could be governed by similar rules.

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APPENDIX: LOCALIZED *PULSES* AS SOLUTIONS TO THE SCHRÖDINGER EQUATION (*APPROXIMATE METHOD*)

As we already said, localized (non-dispersive, besides non-diffracting) *pulses* can be constructed, as solutions to the Schrödinger equation, both by having recourse to the standard "paraxial approximation," and in an exact, analytic way. After having dealt with the exact approach in Sec. III, let us now present also the approximate method, since it can yield simpler (further) equations.

Let us then go back to our Bessel beam solution (4), with condition (5). We can obtain localized (non-dispersive) pulses, as solutions to Schrödinger's equation, by suitably superposing the beam solutions (4), and by selecting in the plane (p_z, E) the straight-line

$$E \;=\; V p_z \; ; \qquad\qquad (p_z \geq 0) \, , \tag{A1}$$

with V a chosen constant speed. Such a straight-line corresponds to the line in Fig. 2 with the simplifying choice $b = 0$: it intersects the parabola $E = p_z^2/(2m)$ at the point corresponding to the value $E = 2mV^2$ (besides at the origin), and it is represented in Fig. 1. From Eq. (5) one gets therefore the important condition

$$E \;\leq\; 2mV^2 \tag{A2}$$

and Eq. (4) can consequently be written

$$\psi(\rho, \zeta) \;=\; J_0(\rho p_\rho/\hbar) \; \exp{[i p_z \zeta/\hbar]}, \tag{A3}$$

where now $p_\rho^2 = (2mE - p_z^2) = E(2m - E/V^2)$ and we introduced the new variable

$$\zeta \;\equiv\; z - Vt \; , \tag{A4}$$

putting explicitly, this time, $p_z > 0$.

Localized-wave solutions can be therefore obtained through the superposition (see Fig. 1):

$$\Psi(\rho, \zeta) \;=\; \mathcal{N} \int_0^{2mV^2} \mathrm{d}E \; J_0\left(\rho \sqrt{\frac{E}{\hbar^2}(2m - \frac{E}{V^2})}\right) \; \exp{[i \frac{E}{\hbar V} \zeta]} \; S(E), \tag{A5}$$

the weight-function $S(E)$ being a suitable energy-spectrum (with the dimensions, as usual, of the inverse of an Energy), while $\mathcal{N}$ is a "normalization" constant which normalizes to 1 the peak-value of $|\Psi|^2$ and (since it multiplies a dimensionless integral) bears the dimensions $[\mathcal{N}] = [L^{3/2}]$, to respect the ordinary meaning of $|\Psi(\rho, \zeta)|^2$. It should be noted that we are integrating, in the space (p_z, E) along the straight-line (A1), that is, $E = Vp_z$. This corresponds to superposing Bessel beams all endowed with the same phase-velocity $V_{\rm ph} \equiv V$. The resulting pulse will possess V as its *group-velocity* (namely, as its peak-velocity), since it is well known that when the phase-velocity $V_{\rm ph}$ does not depend on the energy or frequency, the resulting pulse happens to travel with the group-velocity $V_{\rm g} \equiv \partial\omega/\partial k_z = V_{\rm ph} \equiv V$: cf. Refs. 2, 23, and 31, and references therein. Due to constraint (A2), we are actually integrating along our straight-line from 0 to $2\,mV^2$ (see Fig. 1).

It is important also to note explicitly that each solution $\Psi(\rho, \zeta)$ given by Eq. (A5), depending on z (and t) only via the variable $\zeta \equiv z - Vt$, does represent a pulse that appear with a constant shape to an observer traveling with speed V along the wave motion-line z: in other words, it represents a pulse which propagates rigidly along z. *Therefore, Eq. (A5) is already—as desired—a non-dispersing and non-diffracting ("localized") solutions to the Schrödinger equation.*

Integrals (A5), however, appear difficult to be analytically performed, independently of the spectrum $S(E)$ chosen. To overcome this difficulty, let us go back to Eq. (A3) and rewrite it as a function of p_ρ only, by exploiting Eq. (5b), which can be written $E^2/V^2 - 2mE + p_\rho^2 = 0$, and yields

$$E \;=\; mV^2 \left(1 + \sqrt{1 - \frac{p_\rho^2}{p_{\rho\max}^2}}\right) \; , \tag{A6}$$

where

$$p_{\rho\max} \;=\; mV \; ,$$

as it comes by derivating $E^2/V^2 - 2mE + p_\rho^2 = 0$ with respect to E.

Therefore, Eq. (A3) becomes

$$\psi(\rho, \zeta) \;=\; J_0(\rho p_\rho/\hbar) \; \exp{[i \tfrac{mV}{\hbar} \, \zeta \, \sqrt{1 - \tfrac{p_\rho^2}{m^2V^2}} \;]} \; S(p_\rho/\hbar) \; \mathrm{e}^{i \frac{mV}{\hbar} \zeta} \tag{A7}$$

with $0 \leq p_\rho \leq p_{\rho\max}$, with $p_{\rho\max} = mV$. [For the sake of clarity, let us repeat that when the phase-velocity V becomes (as in our case) the group-velocity, $V_{\rm g} = V$, then the component p_ρ of $\mathbf{p}$ acquires

mV as its maximum value. It holds, moreover, $\sqrt{p^2 - p_\rho^2} = p_z$, which just equals p, since in the present case $V \equiv |\boldsymbol{V}| = V_z$].

Then, the localized solutions will be written as $[p_\rho \geq 0]$:

$$\Psi(\rho, \zeta) \; = \; \mathcal{N} \mathrm{e}^{imV\zeta/\hbar} \int_0^{mV} \mathrm{d}p_\rho \; J_0(\frac{\rho p_\rho}{\hbar}) \; S(p_\rho) \; \exp\left[\frac{imV}{\hbar} \, \zeta \sqrt{1 - \frac{p_\rho^2}{m^2 V^2}} \right] . \tag{A8}$$

Let us notice that, in the new variable p_ρ, the Bessel function, previously written as in Eq. (A5), gets, as we have seen, the simplified expression $J_0(\rho p_\rho)$.

It is now enough to choose a weight-function S that is strongly bumped around the value p_ρ, in the interval $[0, mV]$, with

$$p_\rho \ll mV \; , \tag{A9}$$

for being able to integrate from 0 to ∞ with a negligible error. Namely, let us now adopt the so-called *paraxial approximation*. Under condition (A9), one can approximate the exponential factor as follows:

$$mV \sqrt{1 - \frac{p_\rho^2}{m^2 V^2}} \simeq mV - \frac{1}{2} \frac{p_\rho^2}{mV} \; ,$$

so that Eq. (A8) can be eventually written in terms of an integration from 0 to ∞:

$$\Psi(\rho, \zeta) \; = \; \mathcal{N} \mathrm{e}^{2imV\zeta/\hbar} \int_0^\infty \mathrm{d}p_\rho \; J_0(\frac{\rho p_\rho}{\hbar}) \; S(p_\rho) \; \exp\,[i \frac{p_\rho^2}{2\hbar m V} \zeta] \; . \tag{A10}$$

Let us now examine various special cases of weight-functions $S(p_\rho)$ obeying the previous conditions: that is, well localized around a value $p_\rho \ll mV$.

1. Some examples of approximate localized solutions to the Schrödinger equation (*paraxial approximation*)

As already claimed, we are for the moment adopting the *paraxial approximation,* since it yields good, and interesting enough, results: Only in the following we shall go on to the exact, analytic approach.

First of all, let us consider the simple spectrum

$$S(p_\rho) \; = \; 4q \; p_\rho \, \mathrm{e}^{-q p_\rho^2} \tag{A11}$$

(with the dimensions, now, of the inverse of an impulse), with

$$q \equiv \frac{\alpha}{m^2 V^2} \tag{A12a}$$

so that the above conditions merely imply the dimensionless constant a to be

$$\alpha \gg 1. \tag{A12b}$$

In this case, also the total spectral-width Δp_ρ results to be $\Delta p_\rho \ll mV$: and this too supports the fact that our integral can indeed run till ∞. In Eq. (A6), one can then perform (analytically) the integration, and get the solutions

$$\Psi(\rho, \zeta) \simeq \mathcal{N} \; 4q\hbar^2 \; \mathrm{e}^{2imV\zeta/\hbar} \; \frac{1}{2Q} \; \exp\,[-\frac{\rho^2}{4\hbar(q\hbar - i\frac{1}{mV}\zeta)}] \; , \tag{A13}$$

quantity q being still the one defined in Eq. (A12a), with $\alpha \gg 1$; while function Q is

$$Q \; \equiv \; \hbar(q\hbar - \frac{i}{2mV} \, \zeta) \; . \tag{A14}$$

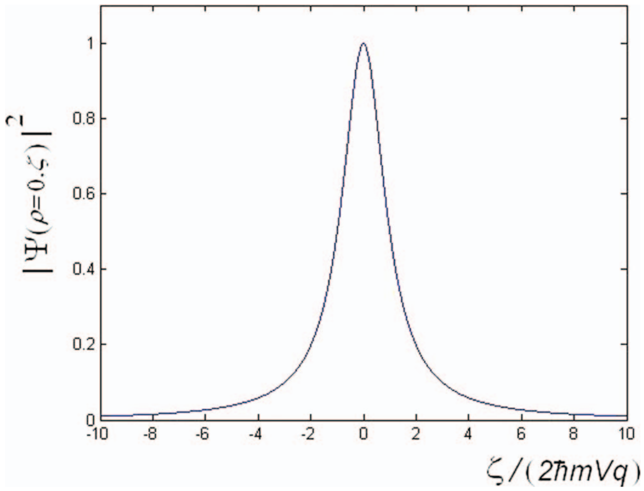

FIG. 9. Behavior of $|\Psi(\rho = 0, \zeta)|^2$ in Eq. (A17), as a function of $\zeta/(2\hbar q m V)$.

Equation (A13) constitutes an interesting solution of the Schrödinger equation: It describes a wavepacket rigidly moving with the chosen speed V. The maximum of its intensity $|\Psi|^2$ occurs at

$$\rho \;=\; 0; \qquad \zeta \;=\; 0\,,$$

and therefore also such a maximum travels with the speed V, as expected (since $\zeta = z - Vt$). For $\zeta = 0$ one gets [$\alpha \gg 1$]:

$$|\Psi(\rho, \zeta = 0)|^2 \;\simeq\; \mathcal{N}^2\; 4\; \exp[-\frac{\rho^2}{2q\hbar^2}]\,, \tag{A15}$$

and the *transverse* localization $\Delta\rho$ of the wavepacket results to be

$$\Delta\rho \;=\; \frac{\hbar}{mV}\,\sqrt{2\alpha}, \tag{A16}$$

which shows also the role of α (and therefore of q) in regulating the wavepacket (constant) transverse total width.

By contrast, putting $\rho = 0$ into Eq. (A13), we end up with the expression [still with $\alpha \gg 1$]:

$$|\Psi(\rho = 0, \zeta)|^2 \;\simeq\; \mathcal{N}^2\; 4\; \frac{q^2\hbar^2}{q^2\hbar^2 + \frac{1}{4m^2V^2}\zeta^2}\,, \tag{A17}$$

which corresponds to

$$\Delta\zeta \;=\; \sqrt{e^2 - 1}\;\frac{2\alpha\hbar}{mV}\,.$$

Solution (A17) is represented in Fig. 9.

Let us briefly consider a few further possible spectra. We shall go on confining ourselves to the simple case of cylindrical symmetry, but analogous solutions can be *easily* found also for more general non-symmetrical cases.

As the second option, let us choose the new spectrum

$$S(p_\rho) \;=\; \frac{1}{p_\rho}\,\mathrm{e}^{-qp_\rho^2}\,, \tag{A18}$$

quantity q being defined in Eq. (A12a), and condition (A12b) being enforced, so that $q \gg 1/(m^2V^2)$ and, again, $\Delta p_\rho \ll mV$. Equation (A6) yields the new solution

$$\Psi(\rho, \zeta) \;\simeq\; \mathcal{N}\,\frac{1}{2}\,\gamma\left(0, \frac{\rho^2}{4Q}\right)\,\exp[\frac{i2mV}{\hbar}\,\zeta]\,, \tag{A19}$$

where function Q is defined in Eq. (17), and γ, here, is the "incomplete gamma function"[29]

$$\gamma(0, \mathcal{A}) \;=\; -\gamma(-1, \mathcal{A}) - \mathcal{A}^{-1}\,\mathrm{e}^{-\mathcal{A}}$$

with

$$\begin{aligned}\gamma(-1, \mathcal{A}) &\equiv -\mathcal{A}^{-1}\,\mathrm{e}^{-\mathcal{A}}\;\Phi(1, 0; \mathcal{A})\\ &\equiv -\mathcal{A}^{-1}\,\mathrm{e}^{-\mathcal{A}}[1 - \Phi(1, 0; \mathcal{A})]\;,\end{aligned}$$

function Φ being the "Probability Integral," that in the present case can be defined as

$$\Phi(1, 0; \mathcal{A}) \;\equiv\; \frac{1}{\Gamma(1)}\int_0^\infty \mathrm{d}x\;\frac{\alpha - \mathrm{e}^{-\mathcal{A}x}}{1 - \mathrm{e}^{-x}}.$$

The maximum, also for solution (A18), occurs at $\rho = \zeta = 0$.

As a third option, we choose

$$S(p_\rho) \;=\; qp_\rho\,\mathrm{e}^{-qp_\rho^2}\,I_0(\frac{sp_\rho}{\hbar}) \tag{A20}$$

always with $\alpha \gg 1$, quantity q being given by Eq. (A12a), s is a constant with the appropriate dimensions of a length (regulating the spectrum bandwidth), and I_0 being the modified Bessel function; one gets from Eq. (A6) the further new solution

$$\Psi(\rho, \zeta) \;\simeq\; \mathcal{N}\,\frac{q\hbar}{2Q}\,\mathrm{e}^{\frac{i2mV}{\hbar}\zeta}\;\exp\left[\frac{s^2 - \rho^2}{4Q}\right]\,J_0\left(\frac{s\rho}{2Q}\right)\;. \tag{A21}$$

As the last option, let us choose

$$S(p_\rho) \;=\; qp_\rho\,\mathrm{e}^{-qp_\rho^2}\,J_0(sp_\rho)\;, \tag{A22}$$

and from Eq. (A6) it follows the fourth solution

$$\Psi(\rho, \zeta) \simeq \mathcal{N}\,\frac{q}{2Q}\,\mathrm{e}^{\frac{i2mV}{\hbar}\zeta}\;\exp\left[-\frac{s^2 + \rho^2}{4Q}\right]\,I_0\left(\frac{s\rho}{2Q}\right)\;. \tag{A23}$$

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